HARDY-LITTLEWOOD CONSTANTS EMBEDDED INTO INFINITE PRODUCTS OVER ALL POSITIVE INTEGERS

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Abstract. A group of infinite products over low-order rational polynomials evaluated at the sequence of prime numbers is loosely called the Hardy-Littlewood constants. In this manuscript we look at them as factors embedded in a super-product over primes, semiprimes, 3-almost primes etc. Numerical tables are derived by transformation into series over k-almost prime zeta functions. Alternative product representations in a basis of k-almost prime products associated with Euler's formula for the Riemann zeta function are also pointed out.

1. Appetizer

As pointed out by J. Vos Post [16], infinite products over rational polynomials evaluated at integers n can be reordered as the integers are individually classified as k-almost primes [4][3, §1.2],

$$\prod_{n=2}^{\infty}\frac{n^2-1}{n^2+1}=\prod_{k=1}^{\infty}\prod_{\substack{n\ge 2\\ \Omega(n)=k}}\frac{n^2-1}{n^2+1}. \tag{1}$$

This classification uses

Definition 1. *(Big-Omega, number of prime factors counted with multiplicity)*

$$\Omega(n)\equiv\sum_j e_j \tag{2}$$

where $n=2^{e_1}3^{e_2}5^{e_3}7^{e_4}11^{e_5}\cdots$ *is the prime factorization of* n.

One approach to numerical practise is the logarithm of the individual terms, [10, 1.513.1]

$$\log\prod_{n=2,\Omega(n)=k}^{\infty}\frac{n^s-1}{n^s+1}=\sum_{n=2,\Omega(n)=k}^{\infty}\log\frac{n^s-1}{n^s+1}=-\sum_{n=2,\Omega(n)=k}^{\infty}\log\frac{1+1/n^s}{1-1/n^s} \tag{3}$$

$$=-2\sum_{l=1}^{\infty}\frac{1}{2l-1}\sum_{n=2,\Omega(n)=k}^{\infty}\frac{1}{n^{s(2l-1)}}=-2\sum_{l=1}^{\infty}\frac{1}{2l-1}P_k(s(2l-1)),$$

supposed we know how to compute the k-almost prime zeta functions of

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TABLE 1. Products emerging from (1). More follow inserting items of Table 2 into (18).

k	s	$\prod_{\Omega(n)=k}(n^s-1)/(n^s+1)$
1	2	.4 = 2/5
2	2	.754499701709514078355718168950541987025077644358722338909979 ...
3	2	.925857274712893127998882138207158415278450218191966021532765 ...
4	2	.980180131878250176004512771774827343256288810830408969770989 ...
1	3	.704072487320784478296298199978624458092583781119988293242884 ...
2	3	.953501120267507619195724656948780658097471099446350601963684 ...
3	3	.993919830234581988800555498147896650698380465169147914350777 ...
1	4	.857142857142857142857142857142857142857142857142857 ... = 6/7
2	4	.990060339240150340510303763520931737419783647516969651298673 ...
3	4	.999371342684423825202796134016609821126691137699898568119928 ...
1	5	.930967939958520284033728872221718554673617521060811182585725 ...
2	5	.997730553624228952992631533134776646504723190210043449878780 ...
3	5	.999928848024661518528748225035933938760888742676503394739919 ...
1	6	.966433566433566433566433566433566433566433566433566 ...= 691/715
2	6	.999462730036393010562666801735890308203188619330023373330279 ...
1	7	.983568510511728007936144959029969386664664363392162577192746 ...
2	7	.999870142148595354702394013511164698444556351264029248220202 ...
1	8	.991910050733580145344851227204168380638968874262991 9 ... = 7234/7293
2	8	.999968223465111892640840105738664881674415326754216354245125 ...

Definition 2. *(k-almost-prime Zeta Function,* [11]*)*

$$P_k(s) \equiv \sum_{\Omega(n)=k} \frac{1}{n^s}. \tag{4}$$

Examples of this factorization of (1) are gathered in Table 1. The particular case of $k=1$, product over the primes, results in rational numbers if s is even (chapter 7 of Ramanujan's first notebook),

$$\prod_p \frac{p^s-1}{p^s+1} = \prod_p \frac{1-p^{-s}}{1+p^{-s}} = \frac{\zeta(2s)}{\zeta^2(s)} = \frac{2B_{2s}}{\binom{2s}{s}B_s^2}, \tag{5}$$

with $B_.$ the Bernoulli numbers, $\zeta(.)$ Riemann's zeta function.

Proof. This follows from Euler's formula

$$\frac{1}{\zeta(s)} = \prod_p \left(1-\frac{1}{p^s}\right), \tag{6}$$

the sign-switched sibling [9, 14]

$$\prod_p \frac{1}{1+p^{-s}} = \prod_p \frac{1-p^{-s}}{(1+p^{-s})(1-p^{-s})} = \prod_p \frac{1-p^{-s}}{1-p^{-2s}} = \frac{\zeta(2s)}{\zeta(s)}, \tag{7}$$

and (again Euler's) [1, 23.2.16]

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}\left|B_{2n}\right|, \quad n=1,2,3,\ldots. \tag{8}$$

□

Remark 1. *The numerical approach to infinite products of rational polynomials as (1) is factorization of numerator and denominator over* $\mathbb{C}$ *with root sets* α_i *and* β_i*, use of the limit of* Γ*-function ratios* [1, (6.1.47)]*—which ought evaluate to unity in the sense* $\sum(\alpha_i - \beta_i) = 0$*, easily verified by subtraction of penultimate coefficients related by the law of Vieta—to end up with a product over* Γ*-ratios* [8, (§1.3)]*:*

$$\prod_{n=1}^{\infty} \frac{\prod_i (n-\alpha_i)}{\prod_i (n-\beta_i)} = \frac{\prod_i \Gamma(1-\beta_i)}{\prod_i \Gamma(1-\alpha_i)} \lim_{n\to\infty} \prod_i \frac{\Gamma(n-\alpha_i)}{\Gamma(n-\beta_i)} = \frac{\prod_i \Gamma(1-\beta_i)}{\prod_i \Gamma(1-\alpha_i)}. \tag{9}$$

For lucky roots of the polynomials, further simplification may be possible by use of the functional equations of the Gamma-function [8, (§1.2)][7].

2. Almost-Prime Zeta Functions of the 2nd Kind

Formulas like (5) and (7) indicate that the products defined in (1) might not represent the atomic constituents of this arithmetic. A vague hope to establish some useful arithmetic basis—plus a glimpse at Euler's formula (6)—proposes

Definition 3. *(k-almost prime zeta functions of the 2nd kind)*

$$\frac{1}{\zeta_k(s)} \equiv \prod_{n\ge 2, \Omega(n)=k} \left(1 - \frac{1}{n^s}\right), \quad \zeta_1(s) = \zeta(s), \quad \Re s > 1. \tag{10}$$

With reference to (4) there are logarithms, the logarithmic derivative,

$$\log \zeta_k(s) = \sum_{j=1}^{\infty} \frac{1}{j} P_k(js), \quad \frac{\zeta_k'(s)}{\zeta_k(s)} = \sum_{j=1}^{\infty} P_k'(js), \tag{11}$$

and their Möbius inversion

$$P_k(s) = \sum_{j=1}^{\infty} \frac{\mu(j)}{j} \log \zeta_k(js). \tag{12}$$

This definition factorizes each of the constants of (1),

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^s}\right) = \prod_{k=1}^{\infty} \frac{1}{\zeta_k(s)}. \tag{13}$$

The unrestricted host product over all integers is evaluated with (9),

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^s}\right) = \prod_{n=1}^{\infty} \frac{(n+1)^s - 1}{(n+1)^s} = \frac{1}{\prod_{l=1}^{s-1} \Gamma(2 - e^{2\pi i l/s})} \tag{14}$$

$$= \begin{cases} 1/2, & s=2 \\ .809396597366290\ldots, & s=3, \\ .919019477593744\ldots, & s=4, \\ .963256561757559\ldots, & s=5, \\ .982684277742192\ldots, & s=6, \\ .991654953472834\ldots, & s=7, \\ .995923315077783\ldots, & s=8, \\ .997991715347709\ldots, & s=9, \\ .999005442480989\ldots, & s=10. \end{cases} = \begin{cases} \frac{\cosh(\pi\sqrt{3}/2)}{3\pi}, & s=3, \\ \frac{\sinh(\pi)}{4\pi}, & s=4, \\ \frac{\cosh^2(\pi\sqrt{3}/2)}{6\pi^2}, & s=6, \\ \frac{\sinh\pi\left[(\sin\tau\cosh\tau)^2 + (\cos\tau\sinh\tau)^2\right]}{8\pi^3}, & s=8, \end{cases} \tag{15}$$

Table 2. ζ_k defined in (10). The first block is the familiar Riemann zeta function [1, Table 23.2]. As $s \to \infty$, $\zeta_k \sim 1 + 2^{-ks}$.

k	s	$\zeta_k(s)$
1	2	1.6449340668482264364724151666460251892189499012067984377355 ...
1	3	1.2020569031595942853997381615114499907649862923404988817922 ...
1	4	1.0823232337111381915160036965411679027747509519187269076829 ...
1	5	1.0369277551433699263313654864570341680570809195019128119741 ...
1	6	1.0173430619844491397145179297909205279018174900328535618424 ...
1	7	1.0083492773819228268397975498497967595998635605652387064172 ...
1	8	1.0040773561979443393786852385086524652589607906498500203291 ...
2	2	1.1541354291311922127531364760826530620213770197691663116011 ...
2	3	1.0242306118269861511158175158755009852679023950490214554774 ...
2	4	1.0050151728999171798274016986522912941640766278579605247722 ...
2	5	1.0011371440971532694447332105947862720228305615568058397033 ...
2	6	1.0002687733197966260458293520190491309253534487772756227744 ...
2	7	1.0000649371192087103992475245979161213426425453760826968155 ...
2	8	1.0000158887627043741809695219326910562197188495738143294511 ...
2	9	1.0000039175994358012012880010835723321071279738093905341977 ...
2	10	1.0000009706055195404576984476915884361069856641058677460911 ...
2	11	1.0000002412172327847094729716543636312861023792401425308611 ...
2	12	1.0000000600686083085780859126280994224248251429993234708244 ...
3	2	1.0394324290304094441498069205216734106792127067461482802922 ...
3	3	1.0030561257336927159342713908343722756666747384706405443733 ...
3	4	1.0003145079830586184568270943970666304250500308241691551611 ...
3	5	1.0000355783602054924193276619405920894891748430962203403255 ...
3	6	1.0000042012916919770985510614241558729211332198227415249922 ...
3	7	1.0000005073890422513287314052576718689678530540360471359255 ...
3	8	1.0000000620681399469078137801483720108089957265766670409511 ...
3	9	1.0000000076516647850564578773513005021687058817988669858211 ...
3	10	1.0000000009478609368564173901584819658760082624860225236388 ...
3	11	1.0000000001177820044320390228148991844747179801698457554066 ...
3	12	1.0000000000146651933786398718879010013321089890457822217644 ...
4	2	1.0100696591819751910787410604390354278765881037871306760677 ...
4	3	1.0003840110773161924205614164881165782735027999641524535033 ...
4	4	1.0000196799485041246608552969870733877791965560225762635422 ...
4	5	1.0000011121065796072645792741438751692254495180830262680977 ...
4	6	1.0000000656486736077825403543041274739769724478846345382155 ...
4	7	1.0000000039640201086333848229956989644559865780369711798188 ...
4	8	1.0000000002424542067763513878454381377821683999848627437466 ...
4	9	1.0000000000149446644057522691729691672153763387828968270533 ...
4	10	1.0000000000009256455280578906353868341456596636528922819400 ...
4	11	1.0000000000000575107453641429481586050838852701903811573055 ...
4	12	1.0000000000000035803694897193423073085083503055044461328477 ...

where $\tau \equiv \pi/\sqrt{2}$.

Cyclotomic factorizations (avoiding divergent expressions, ie, using only polynomial orders of 2 or higher) generalize (7) if $x = 1/n$ is set to visualize the structure:

- From $1-x^4=(1-x^2)(1+x^2)$ we generate

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{1}{n^4}\right)=\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{1}{n^2}\right)\left(1+\frac{1}{n^2}\right). \tag{16}$$

Division leads to

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{1}{n^2}\right)=\frac{\zeta_k(2)}{\zeta_k(4)}, \tag{17}$$

and in general to

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{1}{n^s}\right)=\frac{\zeta_k(s)}{\zeta_k(2s)},\quad \therefore \prod_{n\ge 2,\Omega(n)=k}\frac{n^s-1}{n^s+1}=\frac{\zeta_k(2s)}{\zeta_k^2(s)}. \tag{18}$$

In that sense, the ζ_k keep up to the promise to generate the rational polynomials of Section 1.

- From $1-x^6=(1-x^3)(1+x^3)=(1-x^2)(1+x^2+x^4)$ we generate

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{1}{n^2}+\frac{1}{n^4}\right)=\frac{\zeta_k(2)}{\zeta_k(6)}, \tag{19}$$

and generally from $1-x^{2s}=(1-x^2)(1+x^2+x^4+\cdots+x^{2s-2})$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\sum_{j=1}^{s}\frac{1}{n^{2j}}\right)=\frac{\zeta_k(2)}{\zeta_k(2s+2)}. \tag{20}$$

- The three different ways of grouping factors on the right-hand side of $(1-x^8)=(1-x^2)(1+x^2)(1+x^4)$ establish in addition

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{1}{n^2}+\frac{1}{n^4}-\frac{1}{n^6}\right)=\frac{\zeta_k(4)}{\zeta_k(2)\zeta_k(8)}, \tag{21}$$

a special case of

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\sum_{j=1}^{s}\left(-\frac{1}{n^2}\right)^j\right)=\begin{cases}\dfrac{\zeta_k(4)}{\zeta_k(2)\zeta_k(2s+2)}, & s\,\text{odd};\\[2mm] \dfrac{\zeta_k(4)\zeta_k(2s+2)}{\zeta_k(2)\zeta_k(4s+4)}, & s\,\text{even}.\end{cases} \tag{22}$$

- From $1-x^9=(1-x^3)(1+x^3+x^6)$ by division through $1-x^3$,

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{1}{n^{3j}}+\frac{1}{n^{6j}}\right)=\frac{\zeta_k(3j)}{\zeta_k(9j)}. \tag{23}$$

Other formats are products of these in disguise. The elementary examples are

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{1\mp 2n^s}{n^{2s}}\right)=\prod_{n\ge 2,\Omega(n)=k}\left(1\pm\frac{1}{n^s}\right)^2, \tag{24}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{n^{s+1}\pm n^s\mp 1}{n^{2s+1}}\right)=\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{1}{n^s}\right)\left(1\pm\frac{1}{n^{s+1}}\right), \tag{25}$$

to be continued in Appendix A.

Table 3. Artin's constants defined in (26). Where the k-column is empty, the value is $A^{(r)}$, else $A_k^{(r)}$.

r	k	$A^{(r)}$, $A_k^{(r)}$
1		0.29667513474359103457015502021914286486483151917894789081 6 …
1	1	0.373955813619202288054728054346416415111629248606150042094 …
1	2	0.839042154274468600768462111194541254928307166760882733000 …
1	3	0.958752116435730927714740256578928612659490448502359901592 …
1	4	0.989628867166427665504322837457924308057557589350296534844 …
2		0.673917363376357541664408979322634438564759812312671736792 …
2	1	0.697501358496365903284670350820922924073153946214515395354 …
2	2	0.969932325001525316214920207789129575961145794796696088006 …
2	3	0.996598927480241273419159046329894692291010391011783820658 …
2	4	0.999595278586535535637452493248336453083650632412674049887 …
3		0.850670630791104353750309521250006234999150598195442830656 …
3	1	0.856540444853542174426167984135953882166572800317652140325 …
3	2	0.993521589710505460675409269241416429401115078677815660188 …
3	3	0.999645238332613367730206639120726777503960574831358345008 …
4		0.929838473954346852238318469534553548944908305482253635236 …
4	1	0.931265184160004334389237205550676982558423734587801059016 …
4	2	0.998509500607573754587150213578131277848209932996862159036 …
4	3	0.999959643476392507167505941218783984697629873351553896969 …
5		0.966321276366930291670339804179360258225974383645878751173 …
5	1	0.966668868596777512740328372930016264211423822118193979007 …
5	2	0.999645271339337636005026101684121966092871540313156977621 …
5	3	0.999995220535319372944786385639468021407166618867177761801 …

3. Artin's Constant

Definition 4. *(Artin's Constants of order r)*

$$A^{(r)} \equiv \prod_{n=2}^{\infty}\left(1-\frac{1}{n^r(n-1)}\right) = \prod_{k=1}^{\infty} A_k^{(r)}; \quad A_k^{(r)} \equiv \prod_{\substack{n=2\\ \Omega(n)=k}}^{\infty}\left(1-\frac{1}{n^r(n-1)}\right). \tag{26}$$

$$\begin{aligned} \log A^{(r)} &= \sum_{n=2}^{\infty}\log\left(1-\frac{1}{n^r(n-1)}\right) = -\sum_{n=2}^{\infty}\sum_{s=1}^{\infty}\frac{1}{sn^{rs}(n-1)^s} \\ &= -\sum_{n=2}^{\infty}\sum_{s=1}^{\infty}\sum_{l=0}^{\infty}\binom{-s}{l}\frac{(-1)^l}{sn^{(r+1)s+l}} = -\sum_{s=1}^{\infty}\sum_{l=0}^{\infty}\frac{(s)_l}{sl!}\left[\zeta([r+1]s+l)-1\right] \\ &= \sum_{s=2}^{\infty}\sum_{j=1}^{\lfloor s/(1+r)\rfloor}\frac{1}{j}\binom{s-rj-1}{j-1}[1-\zeta(s)]. \end{aligned} \tag{27}$$

This demonstrates how

$$\log A_k^{(r)} = -\sum_{s=2}^{\infty}\sum_{j=1}^{\lfloor s/(1+r)\rfloor}\frac{1}{j}\binom{s-rj-1}{j-1}P_k(s) \tag{28}$$

is derived; the actual numerical evaluation of $A^{(r)}$ is done easier via (9). Occasionally these Γ-functions simplify:

$$A^{(1)} = -\frac{\sin(\pi\phi)}{\pi}, \quad \phi \equiv \frac{\sqrt{5}+1}{2}. \tag{29}$$

Definition 5. *(Decremented Generalized Lucas Sequences)*

$$a_{r,s} \equiv s \sum_{j=1}^{\lfloor s/(r+1)\rfloor} \frac{1}{j}\binom{s-jr-1}{j-1}. \tag{30}$$

The attribute "decremented" stresses that $1+a_{r,s}$ *are the reference values that one would find, for example, in the Online Encyclopedia of Integer Sequences* [15].

These integer sequences provide the notational shortcut

$$\log A_k^{(r)} = -\sum_{s=2}^{\infty} \frac{a_{r,s}}{s} P_k(s), \tag{31}$$

and have recurrences

$$a_{r,s} = 0, \quad s \le 1; \quad a_{r,s} = 2a_{r,s-1} - a_{r,s-2} + a_{r,s-r-1} - a_{r,s-r-2}, \quad r \ge 1 \tag{32}$$

and generating functions

$$\sum_{s=0}^{\infty} a_{r,s} x^s = \frac{x^{1+r}(1+r-rx)}{(1-x)(1-x-x^{1+r})} = -r - \frac{1}{1-x} + \frac{1+r-rx}{1-x-x^{1+r}}. \tag{33}$$

Values for $s \ge 2$ are

$$\begin{aligned}
a_{1,s} &= 2,3,6,10,17,28,46,75,122,198,321,520,842,1363,2206,3570,\ldots\\
a_{2,s} &= 0,3,4,5,9,14,20,30,45,66,97,143,210,308,452,663,972,1425,2089,\ldots\\
a_{3,s} &= 0,0,4,5,6,7,12,18,25,33,46,65,91,125,172,238,330,456,629,868,\ldots
\end{aligned}$$

The argument of the product (26) admits an exponential product expansion [12, 13]

$$1 - \frac{1}{n^r(n-1)} = \prod_{j=1}^{\infty} \left(1 - \frac{1}{n^j}\right)^{\gamma_{r,j}^{(A)}}. \tag{34}$$

From the Laurent expansion

$$\frac{1}{n^r(n-1)} = \sum_{j=0}^{\infty} \frac{1}{n^{r+1+j}}. \tag{35}$$

we see that this is (up to a sign flip) the inverse Euler transform of the all-1 sequence padded with a short string of initial zeros [6, 2].

$$\gamma_{1,j}^{(A)} = 0,1,1,1,2,2,4,5,8,11,18,25,40,58,90,135,210,316,492,\ldots \tag{36}$$

$$\gamma_{2,j}^{(A)} = 0,0,1,1,1,1,2,2,3,4,6,7,11,14,20,27,39,52,75,102,145,\ldots \tag{37}$$

$$\gamma_{3,j}^{(A)} = 0,0,0,1,1,1,1,1,2,2,3,3,5,6,8,10,14,17,24,30,41,53,\ldots \tag{38}$$

Whereas (28) expands $A_k^{(r)}$ in the P_k basis, (34) generates them in ζ_k basis:

$$A_k^{(r)} = \prod_{\Omega(n)=k}\prod_j \left(1-\frac{1}{n^j}\right)^{\gamma_{r,j}^{(A)}} = \prod_j \zeta_k(j)^{-\gamma_{r,j}^{(A)}}. \tag{39}$$

The same result could be obtained by plugging (12) into the right hand side of (31), then exponentiation, which reveals the Möbius pair

$$\frac{1}{j}\sum_{l|j}\mu(l)a_{r,j/l} = \gamma_{r,j}^{(A)}; \quad a_{r,s} = \sum_{l|s} l\gamma_{r,l}^{(A)}. \tag{40}$$

Remark 2. *This connection between the two types of coefficients does not depend on the mediation by P nor on summation over n. Supposed any γ_j are defined in*

$$X \equiv \prod_j \left(1-\frac{1}{n^j}\right)^{\gamma_j}, \tag{41}$$

which implies

$$\log X = \sum_j \log\left(1-\frac{1}{n^j}\right)^{\gamma_j} = \sum_j \gamma_j \log\left(1-\frac{1}{n^j}\right) = -\sum_j \gamma_j \sum_{s\ge 1}\frac{1}{jn^{sj}}. \tag{42}$$

Let furthermore g_k be defined via

$$\log X \equiv -\sum_k \frac{g_k}{k}\frac{1}{n^k}, \tag{43}$$

then—by comparison of coefficients of equal powers $k = sj$ of n—

$$\frac{g_k}{k} = \sum_{j|k}\frac{\gamma_j}{k/j} \quad \therefore \quad g_k = \sum_{j|k} j\gamma_j \quad \therefore \quad \gamma_j = \frac{1}{j}\sum_{l|j}\mu(l)g_{j/l}. \tag{44}$$

4. Twin Prime Constants

Definition 6. *(Twin Prime Constants of order r)*

$$T^{(r)} \equiv \prod_{n=3}^{\infty}\left(1-\frac{1}{(n-1)^r}\right) = \prod_{k=1}^{\infty} T_k^{(r)}; \quad T_k^{(r)} \equiv \prod_{\substack{n=3\\ \Omega(n)=k}}^{\infty}\left(1-\frac{1}{(n-1)^r}\right). \tag{45}$$

$$\begin{aligned}\log T^{(r)} &= \sum_{n=3}^{\infty}\log\left(1-\frac{1}{(n-1)^r}\right) = -\sum_{n=3}^{\infty}\sum_{s=1}^{\infty}\frac{1}{s(n-1)^{rs}}\\
&= -\sum_{n=3}^{\infty}\sum_{s=1}^{\infty}\sum_{l=0}^{\infty}\binom{-rs}{l}\frac{(-1)^l}{sn^{rs+l}} = -\sum_{n=3}^{\infty}\sum_{s=1}^{\infty}\sum_{l=0}^{\infty}\frac{(rs)_l}{sl!n^{rs+l}}\\
&= -\sum_{s=1}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(rs+l)}{\Gamma(rs)sl!}\left[\zeta(rs+l)-1-\frac{1}{2^{rs+l}}\right]\end{aligned} \tag{46}$$

$$= -\sum_{s=r}^{\infty}\sum_{j=1}^{\lfloor s/r\rfloor}\frac{1}{j}\binom{s-1}{rj-1}\left[\zeta(s)-1-\frac{1}{2^s}\right] = -\sum_{n=2}^{\infty}\sum_{s=1}^{\infty}\frac{1}{sn^{rs}} = \sum_{s=1}^{\infty}\frac{1}{s}\left[1-\zeta(rs)\right].$$

This is a sum rule for the zeta function. The actual values of $T^{(r)}$ duplicate those of (15) because

$$T^{(r)} = \prod_{n=2}^{\infty}(1-\frac{1}{n^r}). \tag{47}$$

Repeating (46) yields

$$\log T_k^{(r)} = -\sum_{s=1}^{\infty}\sum_{j=1}^{\lfloor s/r\rfloor}\frac{1}{j}\binom{s-1}{rj-1}\times\begin{cases} P_k(s), & k>1 \\ P(s)-\frac{1}{2^s}, & k=1 \end{cases}. \tag{48}$$

One can rewrite this as

$$\log T_k^{(r)} = -\sum_{s=1}^{\infty}\frac{r}{s}t_{r,s}\times\begin{cases} P_k(s), & k>1 \\ P(s)-\frac{1}{2^s}, & k=1 \end{cases}. \tag{49}$$

by introducing integer sequences $t_{r,s}$ via

Definition 7. *(Binomial transform of aerated all-1 sequences)*

$$t_{r,s} \equiv \sum_{j=1}^{\lfloor s/r\rfloor}\frac{s}{rj}\binom{s-1}{rj-1}. \tag{50}$$

Generating functions are

$$\sum_{s=1}^{\infty} t_{r,s}x^s = \begin{cases} \frac{x^2}{(1-x)(1-2x)}, & r=2, \\ \frac{x^3}{(1-x)(1-2x)(1-x+x^2)}, & r=3, \\ \frac{x^4}{(1-x)(1-2x)(1-2x+2x^2)}, & r=4, \\ \frac{x^5}{(1-x)(1-2x)(1-3x+4x^2-2x^3+x^4)}, & r=5, \\ \frac{x^6}{(1-x)(1-2x)(1-4x+7x^2-6x^3+3x^4)}, & r=6, \\ \frac{x^7}{(1-x)(1-2x)(1-5x+11x^2-13x^3+9x^4-3x^5+x^6)}, & r=7. \end{cases} \tag{51}$$

$$t_{2,s} = 0,1,3,7,15,31,63,127,255,511,1023,2047,4095,8191,16383\ldots \tag{52}$$
$$t_{3,s} = 0,0,1,4,10,21,42,84,169,340,682,1365,2730,5460,10921\ldots \tag{53}$$
$$t_{4,s} = 0,0,0,1,5,15,35,71,135,255,495,991,2015,4095,8255,16511\ldots \tag{54}$$

The exponents $\gamma_{r,j}^{(T)}$ of the inverse Euler transformation are defined as

$$1-\frac{1}{(n-1)^r} = \prod_{j=1}^{\infty}\left(1-\frac{1}{n^j}\right)^{\gamma_{r,j}^{(T)}}, \quad T_k^{(r)} = \prod_{j\ge 2}\zeta_k^{-\gamma_{r,j}^{(T)}}. \tag{55}$$

Examples at indices $j \ge 2$ are

$$\gamma_{2,j}^{(T)} = 1,2,3,6,9,18,30,56,99,186,335,630,1161,2182,4080,\ldots, \tag{56}$$
$$\gamma_{3,j}^{(T)} = 0,1,3,6,10,18,30,56,99,186,335,630,1161,2182,4080,\ldots, \tag{57}$$
$$\gamma_{4,j}^{(T)} = 0,0,1,4,10,20,35,60,100,180,325,620,1160,2200,4110,\ldots, \tag{58}$$
$$\gamma_{5,j}^{(T)} = 0,0,0,1,5,15,35,70,126,215,355,605,1065,2002,3855,\ldots. \tag{59}$$

Table 4. Constants defined in (45).

r	k	$T_k^{(r)}$
2	1	0.66016181584686957392781211001455577843262336028473341331 94 ...
2	2	0.80450826124747420034776097558048402588429162353844113734 25 ...
2	3	0.95505728822987004930144625580414157341473340174743371811 50 ...
2	4	0.98920513231930153958078167977119571959369254030369584163 22 ...
3	1	0.85539210200339860855096192381733691427796173263437618919 09 ...
3	2	0.95075435135761591085628518481282993705681982577356561383 28 ...
3	3	0.99574720309341137875406873066673476616292253617518653627 05 ...
3	4	0.99954971609067455096627510606570048701416230463415725871 67 ...
4	1	0.93294727880502251542454741177247766912678029333159769226 89 ...
4	2	0.98560131550807644165525583745570030445237582039591268118 51 ...
4	3	0.99948844711802706609314539254943336675900055380610960533 90 ...
4	4	0.99997512687670946226180622557724786101355495829248566889 74 ...
5	1	0.96766416414492739286845882845540670206037350493955563637 90 ...
5	2	0.99551340676039725886312191397269812755679204022581635096 88 ...
5	3	0.99993294788727187501921136192361982385467156034977564480 8 ...
5	4	0.99999849052907649509443046101098420301785483513939001930 5 ...

$\gamma_{2,j}^{(T)}$ and $\gamma_{3,j}^{(T)}$ differ only by one at $j = 2$, 3 and 6, caused by

$$\frac{T_k^{(2)}}{T_k^{(3)}} = \frac{\zeta_k(6)}{\zeta_k(2)\zeta_k(3)}. \tag{60}$$

Equivalent to (40) we have

$$\frac{r}{j}\sum_{l|j}\mu(l)t_{r,j/l} = \gamma_{r,j}^{(T)}; \quad t_{r,s} = \frac{1}{r}\sum_{l|s} l\gamma_{r,l}^{(T)}. \tag{61}$$

5. Quadratic Class Number

A sign flip in a denominator of (26) provides

Definition 8. *(Quadratic Class numbers of order r)*

$$Q^{(r)} \equiv \prod_{n=2}^{\infty}\left(1-\frac{1}{n^r(n+1)}\right) = \prod_{k=1}^{\infty} Q_k^{(r)}; \quad Q_k^{(r)} \equiv \prod_{\substack{n=2\\ \Omega(n)=k}}^{\infty}\left(1-\frac{1}{n^r(n+1)}\right). \tag{62}$$

The special value

$$Q^{(1)} = -\frac{2\sin(\pi\phi)}{\pi} = 2A^{(1)}, \quad \phi \equiv \frac{\sqrt{5}+1}{2}, \tag{63}$$

is found with (9). Essentially duplicating the calculation in (27) we have

$$\log Q^{(r)} = \sum_{n=2}^{\infty}\log\left(1-\frac{1}{n^r(n+1)}\right) = -\sum_{n=2}^{\infty}\sum_{s=1}^{\infty}\frac{1}{sn^{rs}(n+1)^s} \tag{64}$$
$$= \sum_{s=2}^{\infty}\sum_{j=1}^{\lfloor s/(1+r)\rfloor}\frac{(-1)^{s-(r+1)j}}{j}\binom{s-rj-1}{j-1}[1-\zeta(s)].$$

Table 5. Constants defined in (62). Where the k-column is empty, the value is $Q^{(r)}$, else $Q_k^{(r)}$.

r	k	$Q^{(r)}$, $Q_k^{(r)}$
1		0.593350269487182069140310040438285729729663038357895781633
1	1	0.704442200999165592736603350326637210188586431417098049414 ...
1	2	0.884490615792645569156126530213936198197151790687002628832 ...
1	3	0.964758474366761979144138911837762037272106901540505442479 ...
1	4	0.990393442303116742704510324208970590429148773419205301803 ...
2		0.861465028009033072712078741634897482256486581138963373814 ...
2	1	0.881513839725170776928391822903227847129869257208076733670 ...
2	2	0.980376289243855939864938619579944255828568405323127482357 ...
2	3	0.997235390988435066639517992790322243014456604416487473356 ...
2	4	0.999634762421613451087385734159581784386619475630152927741 ...
3		0.943588586975055819450398681669287957384984288942155352763 ...
3	1	0.947733262143675375939521537654189613033631632317413852828 ...
3	2	0.995928002832231110330556534959384975519658530911655791754 ...
3	3	0.999717415940667569960015810241340983047451116331199660541 ...
3	4	0.999981370624803353143270623749614063667365657870395981306 ...
4		0.974894913359018345579696732396272984272954782066491224441 ...
4	1	0.975824153047668241679011436594799831971764971229212609442 ...
4	2	0.999080621322852565861019671107083386967236557968329389061 ...
4	3	0.999968172024482705270548857407825523042346422728579249674 ...
4	4	0.999998949807673405679237367569730617874479395755489593508 ...
5		0.988286601083665561883354705652451815692957697529740408374 ...
5	1	0.988504397741246908751106623851186666440095808327534618120 ...
5	2	0.999783481766640731283537327557402867111772830353712345808 ...
5	3	0.999996250854906273732628391333715607260086058728852979934 ...
5	4	0.999999938086969714053459138631507932333794604460092491194 ...

Table 5 is calculated via

$$\log Q_k^{(r)} = -\sum_{s=2}^{\infty} \sum_{j=1}^{\lfloor s/(1+r) \rfloor} \frac{(-1)^{s-(r+1)j}}{j} \binom{s-rj-1}{j-1} P_k(s) = -\sum_{s=2}^{\infty} \frac{1}{s} q_{r,s} P_k(s). \tag{65}$$

The previous line introduces auxiliary integer sequences $q_{r,s}$ equivalent to (30) with

Definition 9. *(Binomial transforms of aerated alternating-1 sequences)*

$$q_{r,s} \equiv s \sum_{j=1}^{\lfloor s/(r+1) \rfloor} \frac{(-1)^{s-(r+1)j}}{j} \binom{s-jr-1}{j-1}. \tag{66}$$

Lists at $s \geq 2$ are

$$\begin{aligned}
q_{1,s} &= 2,-3,6,-10,17,-28,46,-75,122,-198,321,-520,842,-1363,2206,-3570\ldots \\
q_{2,s} &= 0,3,-4,5,-3,0,4,-6,5,0,-7,13,-14,8,4,-17,24,-19,1,24,-44,46,-23,-20,\ldots \\
q_{3,s} &= 0,0,4,-5,6,-7,12,-18,25,-33,46,-65,91,-125,172,-238,330,-456,629,\ldots \\
q_{4,s} &= 0,0,0,5,-6,7,-8,9,-5,0,6,-13,21,-25,24,-17,3,19,-45,70,-88,92,-74,30,\ldots
\end{aligned}$$

Recurrences and generating function are sign-flipped variants of (32) and (33),

$$q_{r,s}=0,\quad s\le 1;\quad q_{r,s}=-2q_{r,s-1}-q_{r,s-2}+q_{r,s-r-1}+q_{r,s-r-2},\quad r\ge 1, \tag{67}$$

$$\sum_{s=0}^{\infty} q_{r,s}x^s=\frac{x^{1+r}(1+r+rx)}{(1+x)(1+x-x^{1+r})}=-r-\frac{1}{1+x}+\frac{1+r+rx}{1+x-x^{1+r}}. \tag{68}$$

The zeta-expansion exponents $\gamma_{r,j}^{(Q)}$ are defined to satisfy

$$1-\frac{1}{n^r(1+n)}=\prod_{j=1}^{\infty}\left(1-\frac{1}{n^j}\right)^{\gamma_{r,j}^{(Q)}},\quad Q_k^{(r)}=\prod_{j=2}^{\infty}\zeta_k^{-\gamma_{r,j}^{(Q)}}. \tag{69}$$

In the range $j\ge 2$ we find

$$\gamma_{1,j}^{(Q)}=1,-1,1,-2,3,-4,5,-8,13,-18,25,-40,62,-90,135,-210,\ldots \tag{70}$$

$$\gamma_{2,j}^{(Q)}=0,1,-1,1,-1,0,1,-1,0,0,0,1,-1,0,0,-1,2,-1,0,1,-2,2,\ldots \tag{71}$$

$$\gamma_{3,j}^{(Q)}=0,0,1,-1,1,-1,1,-2,3,-3,3,-5,7,-8,10,-14,19,-24,30,\ldots \tag{72}$$

$$\gamma_{4,j}^{(Q)}=0,0,0,1,-1,1,-1,1,-1,0,1,-1,1,-2,2,-1,0,1,-2,3,-4,4.\ldots \tag{73}$$

The analogue of (40) relates $q_{r,j}$ with $\gamma_{r,j}^{(Q)}$,

$$\frac{1}{j}\sum_{l|j}\mu(l)q_{r,j/l}=\gamma_{r,j}^{(Q)};\quad q_{r,s}=\sum_{l|s}l\gamma_{r,l}^{(Q)}. \tag{74}$$

6. Feller-Tornier

Definition 10. *(Feller-Tornier Constants)*

$$F^{(r)}\equiv\prod_{n=2}^{\infty}\left(1-\frac{2}{n^r}\right)=\prod_{k=1}^{\infty}F_k^{(r)};\quad F_k^{(r)}\equiv\prod_{\substack{n=2\\ \Omega(n)=k}}^{\infty}\left(1-\frac{2}{n^r}\right). \tag{75}$$

Closed form expressions generated from the n roots of $(n+1)^r-2$ via (9) are:

$$F^{(2)} = -\frac{\sin(\pi\sqrt{2})}{\pi\sqrt{2}}, \tag{76}$$

$$F^{(4)} = -\frac{\sinh(\pi\sqrt[4]{2})\sin(\pi\sqrt[4]{2})}{\pi^2\sqrt{2}}, \tag{77}$$

$$F^{(6)} = -\frac{\sin(\pi 2^{1/6})\left[\cosh^2(\pi 2^{-5/6}\sqrt{3})-\cos^2(\pi 2^{-5/6})\right]}{\pi^3\sqrt{2}}. \tag{78}$$

Table 6 is produced accumulating

$$\log F_k^{(r)}=\sum_{n\ge 2,\Omega(n)=k}\log\left(1-\frac{2}{n^r}\right)=-\sum_{j=1}^{\infty}\frac{2^j}{j}P_k(rj). \tag{79}$$

The zeta-expansion exponents $\gamma_{r,j}^{(F)}$ are

$$1-\frac{2}{n^r}=\prod_{j=1}^{\infty}\left(1-\frac{1}{n^j}\right)^{\gamma_{r,j}^{(F)}},\quad F_k^{(r)}=\prod_j\zeta_k(j)^{-\gamma_{r,j}^{(F)}}. \tag{80}$$

TABLE 6. Constants defined in (75). Where the k-column is empty, the value is $F^{(r)}$, else $F_k^{(r)}$.

r	k	$F^{(r)}$, $F_k^{(r)}$
2		0.21695429437747636935686403906343759659991329971424145295 0 ...
2	1	0.322634098939244670579531692548237066570950579665832709961 ...
2	2	0.746546589392028395045090198448531702307441861924359721333 ...
2	3	0.925267218004050491882112057709653605835184213438593596030 ...
2	4	0.980141423819685616602097797704982658698681266298832028699 ...
3		0.640575909221546133846815305854929804642156633120487523553 ...
3	1	0.676892737009881993610237326724389212797678397459788845273 ...
3	2	0.952981131498858970286460835642413457907291749293329766096 ...
3	3	0.993911463538908938115880607314708743157809479259098501054 ...
3	4	0.999232354384330965293278269072153569836124255417702962004 ...
4		0.840695833076274061650473710681177939252057653860271740567 ...
4	1	0.849732991384718766265053703629160439892820104242861046497 ...
4	2	0.990028758685057521928151579163269491391556722507312328948 ...
4	3	0.999371218596872044721227851298066869151465994816088393076 ...
4	4	0.999960641022410557951000962330383249199360087979036126331 ...
5		0.926880857710656853256795504739069232524104014194213179034 ...
5	1	0.929059192959662815115245871984200623766376123420999266247 ...
5	2	0.997728614956085550822520064832476818851830182381404991433 ...
5	3	0.999928846129017559729606214395367084797186701036127260135 ...
5	4	0.999997775789625477894415002081234674890918717170135941506 ...

(79) in conjunction with (12) reveals

$$\gamma_{r,j}^{(F)} = \begin{cases} 0 & , r \nmid j, \\ \dfrac{r}{j} \displaystyle\sum_{d|(j/r)} 2^d \mu(\frac{j}{rd}) & , r \mid j, \end{cases} \tag{81}$$

so values in the range $j \geq 2$ start as

$$\gamma_{2,j}^{(F)} = 2, 0, 1, 0, 2, 0, 3, 0, 6, 0, 9, 0, 18, 0, 30, 0, 56, 0, 99, 0, 186, 0, 335, \ldots \tag{82}$$

$$\gamma_{3,j}^{(F)} = 0, 2, 0, 0, 1, 0, 0, 2, 0, 0, 3, 0, 0, 6, 0, 0, 9, 0, 0, 18, 0, 0, 30, 0, 0, 56, 0, \ldots \tag{83}$$

The others are obvious: the count of zero fillers—factors effectively dropping out in (80)—grows simply as $r - 1$.

7. HARDY-LITTLEWOOD

The Hardy-Littlewood constants, in the narrow sense, are the cases $k = 1$ in

Definition 11.

$$C^{(3)} \equiv \prod_{n=4}^{\infty}\left(1-\frac{3n-1}{(n-1)^3}\right)=\frac{2}{9};\; C_k^{(3)} \equiv \prod_{\substack{n\ge 4\\ \Omega(n)=k}}\left(1-\frac{3n-1}{(n-1)^3}\right); \tag{84}$$

$$C^{(4)} \equiv \prod_{n=5}^{\infty}\left(1-\frac{6n^2-4n+1}{(n-1)^4}\right)=\frac{3}{32};\; C_k^{(4)} \equiv \prod_{\substack{n\ge 5\\ \Omega(n)=k}}\left(1-\frac{6n^2-4n+1}{(n-1)^4}\right); \tag{85}$$

$$C^{(r)} \equiv \prod_{n>r}\frac{n^{r-1}(n-r)}{(n-1)^r}=\frac{(r-1)!}{r^{r-1}};\; C_k^{(r)} \equiv \prod_{\substack{n>r\\ \Omega(n)=k}}\frac{n^{r-1}(n-r)}{(n-1)^r},\quad r\ge 3. \tag{86}$$

$T^{(2)} = C^{(2)}$ might be linked in here. As before, we turn to the logarithm and define associate integer expansion coefficients $c_s^{(r)}$ from their Laurent expansion,

$$\log C_k^{(r)} = \sum_{n>r,\Omega(n)=k}\log\frac{n^{r-1}(n-r)}{(n-1)^r} = \sum_{n>r,\Omega(n)=k}\log\frac{1-r/n}{(1-1/n)^r} \tag{87}$$

$$= \sum_{n>r,\Omega(n)=k}\left[\log\left(1-\frac{r}{n}\right)-r\log\left(1-\frac{1}{n}\right)\right] = \sum_{n>r,\Omega(n)=k}\left[-\sum_{s\ge1}\frac{r^s}{sn^s}+r\sum_{s\ge1}\frac{1}{sn^s}\right],$$

which can be summarized after interchange of summations as

$$\log C_k^{(r)} = -\sum_{n>r,\Omega(n)=k}\sum_{s=2}^{\infty}\frac{1}{s}\frac{c_s^{(r)}}{n^s}, \tag{88}$$

in a simple common format:

$$c_s^{(r)} \equiv \begin{cases} 0, & s<2;\\ r^s-r, & s\ge 2.\end{cases} \tag{89}$$

Plugging (4) into the right hand side of (88), we compensate for the fact that the lower limits $n > r$ discard some n for the prime ($k = 1$) and semiprime ($k = 2$) cases:

$$\log C_k^{(r)} = -\sum_{s\ge2}\frac{1}{s}c_s^{(r)}\times\begin{cases} \left[P_k(s)-\frac{1}{2^s}-\frac{1}{3^s}-\frac{1}{5^s}\right], & r=5,6,\quad k=1;\\ \left[P_k(s)-\frac{1}{2^s}-\frac{1}{3^s}\right], & r=3,4,\quad k=1;\\ \left[P_k(s)-\frac{1}{4^s}-\frac{1}{6^s}\right], & r=6,7,8,\quad k=2;\\ \left[P_k(s)-\frac{1}{4^s}\right], & r=4,5,\quad k=2;\\ P_k(s), & r=3,\quad k=2;\\ P_k(s), & 3\le r<8,\ k\ge 3.\end{cases} \tag{90}$$

Remark 3. *The convergence of these series is slow, given that $c_s^{(r)}$ grows roughly $\sim r^s$ and the terms right from the brace fall roughly $\sim r^{-s}$. The standard acceleration technique is to split the products (86) into $\prod_{n>r} = \prod_{r<n\le M}\cdot\prod_{n>M}$ with some free integer M of the order of some tenths of r, to calculate the first of these two products explicitly, and to subtract all of the inverse powers of the k-almost primes below M on the right hand side of (90), so the term to the right of the brace falls off roughly $\propto M^{-s}$* [14].

TABLE 7. Hardy-Littlewood constants from equations (84)–(86).

k	$C_k^{(3)}$
1	0.635166354604271207206696591272522417342065687332372450899 ...
2	0.424234558470737235218539671836177441479432573726566654172 ...
3	0.861978217115406397600389288178363010882226721530095958070 ...
4	0.967010333852598290029706098677367652117366812663687575395 ...
5	0.991986542483777613682589437104065646426247353044705005169 ...
	$C_k^{(4)}$
1	0.307494878758327093123354486071076853022178519950663928298 ...
2	0.461691758364773730232305524418356233105041873484187592372 ...
3	0.723165327592227885742644081506537901793428021559355399358 ...
4	0.933085922756286271428677500179124619333526568552797663388 ...
5	0.983814785274894677909254622541376985294671241029387304408 ...
	$C_k^{(5)}$
1	0.409874885088236474478781212337955277896358013254945469826 ...
2	0.199805231972458892888828284513805175888486651003235386176 ...
3	0.547976628430836989696044385054920027204371593584787789976 ...
4	0.887429384542023666239166084035101114754091610659986923636 ...
5	0.972787328073924092604485187403768369490797977225729870114 ...
	$C_k^{(6)}$
1	0.186614297358358396656924847944188337840073944945589304871 ...
2	0.298042020487754531592316128677284826210605850852999108669 ...
3	0.353138894039211423074594163633660113968205949561584457073 ...
4	0.830410751660277094955031322533872844216167956532800549112 ...
5	0.958867249262078290883709484646892100264447229912351544026 ...

Exponents $\gamma_{r,j}^{(C)}$ are defined with the aim to decompose (86):

$$\frac{n^{r-1}(n-r)}{(n-1)^r} \equiv \prod_{j\geq 1}\left(1-\frac{1}{n^j}\right)^{\gamma_{r,j}^{(C)}}, \tag{91}$$

related to $c_s^{(r)}$ via a Möbius transform as described in (44) [13]:

$$\begin{aligned}
\gamma_{3,j} &= 3, 8, 18, 48, 116, 312, 810, 2184, 5880, 16104, 44220, 122640, 341484, \ldots, \\
\gamma_{4,j} &= 6, 20, 60, 204, 670, 2340, 8160, 29120, 104754, 381300, 1397740, 5162220, \ldots, \\
\gamma_{5,j} &= 10, 40, 150, 624, 2580, 11160, 48750, 217000, 976248, 4438920, 20343700, \ldots
\end{aligned}$$

Eq. (91) rephrases the constants (84)–(86) as

$$C_k^{(r)} = \prod_{j\geq 2} \lambda_k(r,j)^{-\gamma_{r,j}^{(C)}}, \tag{92}$$

where λ_k takes into account that the products may have been defined without the first one to three primes or semiprimes,

(93)
$$\lambda_k(r,j) = \prod_{\substack{n>r\\ \Omega(n)=k}} \frac{1}{(1-1/n^j)} = \zeta_k(j)\times \begin{cases} (1-\frac{1}{2^j})(1-\frac{1}{3^j})(1-\frac{1}{5^j}), & r=5,\ k=1;\\ (1-\frac{1}{2^j})(1-\frac{1}{3^j}), & r=3,4,\ k=1;\\ (1-\frac{1}{4^j}), & r=4,5,\ k=2;\\ 1, & r=3,\ k=2;\\ 1, & r=3,4,5,\ k\ge 3. \end{cases}$$

8. Summary

The familiar Hardy-Littlewood, Artin's, Feller-Tornier and similar constants are infinite products over the prime numbers. We have generalized these to products over k-almost primes, and provide tables for low ranks and small k. The products over all values of k are infinite "host" products, easily evaluated as a multi-gamma functions associated with the roots of the defining rational polynomial.

Appendix A. Hybrids

Hybrids are products and ratios of the constants discussed above, which provide access to other forms of products. One example of such a reduction is, see (18),

(94)
$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{3n^s+2}{n^{3s}}\right) = \prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{2}{n^s}\right)\left(1+\frac{1}{n^s}\right)^2 = F_k^{(s)}\left(\frac{\zeta_k(s)}{\zeta_k(2s)}\right)^2.$$

An—obviously incomplete—sample of these is:

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{2n^s-1}{(n^s-1)^2}\right) = \zeta_k^2(s), \tag{95}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{n^{s+l}+n^s-1}{n^{2s+l}-n^{s+l}-n^s+1}\right) = \zeta_k(s+l)\zeta_k(s), \tag{96}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{1}{n^s+n^{s-1}+\cdots+n^2+n+1}\right) = \frac{\zeta_k(s+1)}{\zeta_k(s)}, \tag{97}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{n^l-1}{n^{s+l}-1}\right) = \frac{\zeta_k(s+l)}{\zeta_k(s)}, \tag{98}$$

(99)
$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{2n^s-1}{n^{2s}}\right) = \frac{1}{\zeta_k^2(s)},\qquad \prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{n^{s+l}+n^s-1}{n^{2s+l}}\right) = \frac{1}{\zeta_k(s+l)\zeta_k(s)},$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{1}{n(n^{s-1}+n^{s-2}+\cdots+n+1)}\right) = \frac{\zeta_k(s)}{\zeta_k(s+1)}, \tag{100}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{1}{n^s+1}\right) = \frac{\zeta_k(2s)}{\zeta_k(s)}, \tag{101}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{n^{(l-1)s}-1}{n^{ls}-1}\right)=\frac{\zeta_k(ls)}{\zeta_k(s)}, \tag{102}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{1}{n^s(n-1)-1}\right)=\frac{1}{A_k^{(s)}}, \tag{103}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{1}{n^{s+2}-n^{s+1}-n+1}\right)=\zeta_k(s+1)A_k^{(s)}, \tag{104}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{2n+1}{n^3-2n-1}\right)=\frac{\zeta_k(2)}{A_k^{(1)}}, \tag{105}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{1}{n(n^{s+1}-n^s-1)}\right)=\frac{1}{\zeta_k(s+1)A_k^{(s)}}, \tag{106}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{2n+1}{n^3}\right)=\frac{A_k^{(1)}}{\zeta_k(2)},\qquad \prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{2n^s+n^{s-1}+n^{s-2}+\cdots+n+1}{n^{2s+1}}\right)=\frac{A_k^{(s)}}{\zeta_k(s+1)}, \tag{107}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{n-2}{n^{s+1}-n^s-n+1}\right)=\zeta_k(s)A_k^{(s)}, \tag{108}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{n-2}{n^{s+1}-n^s-1}\right)=\frac{1}{\zeta_k(s)A_k^{(s)}}, \tag{109}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{n^s+n^{s-1}+\cdots+n+1}{n^{2s}}\right)=\frac{A_k^{(s)}}{\zeta_k(s)}, \tag{110}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{n^{s-1}+n^{s-2}+\cdots+n^2+2n+1}{n^{s+1}}\right)=\frac{A_k^{(1)}}{\zeta_k(s)}, \tag{111}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{1}{n^s(n+1)-1}\right)=\frac{1}{Q_k^{(s)}}, \tag{112}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{1}{n^{s+2}+n^{s+1}-n-1}\right)=\zeta_k(s+1)Q_k^{(s)}, \tag{113}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{2n+1}{n^3-2n+1}\right)=\frac{\zeta_k(2)}{Q_k^{(1)}}, \tag{114}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{1}{n(n^{s+1}+n^s-1}\right)=\frac{1}{\zeta_k(s+1)Q_k^{(s)}}, \tag{115}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{2n-1}{n^3}\right)=\frac{Q_k^{(1)}}{\zeta_k(2)},\qquad \prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{2n^{s+1}+n^s-1}{n^{2s+1}(n+1)}\right)=\frac{Q_k^{(s)}}{\zeta_k(s+1)}, \tag{116}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{n}{n^{s+1}+n^s-n-1}\right)=\zeta_k(s)Q_k^{(s)}, \tag{117}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{n}{n^{s+1}+n^s-1}\right)=\frac{1}{\zeta_k(s)Q_k^{(s)}}, \tag{118}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{n^2+n-1}{n^4}\right)=\frac{Q_k^{(2)}}{\zeta_k(2)},\qquad \prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{n^{s+1}+2n^s-1}{n^{2s}(n+1)}\right)=\frac{Q_k^{(s)}}{\zeta_k(s)}, \tag{119}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{2}{n^s-2}\right)=\frac{1}{F_k^{(s)}}, \tag{120}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{1}{n^s-1}\right)=\zeta_k(s)F_k^{(s)},\qquad \prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{2n^l-1}{n^{s+l}-1}\right)=\zeta_k(s+l)F_k^{(s)}, \tag{121}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{n^l-2}{n^l(n^s-1)}\right)=\zeta_k(s)F_k^{(s+l)}, \tag{122}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{3n^s-2}{n^{2s}-3n^s+2}\right)=\frac{\zeta_k(s)}{F_k^{(s)}}, \tag{123}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{1}{n^s-2}\right)=\frac{1}{\zeta_k(s)F_k^{(s)}},\qquad \prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{2n^l-1}{n^l(n^s-2)}\right)=\frac{1}{\zeta_k(s+l)F_k^{(s)}}, \tag{124}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{3n^s-2}{n^{2s}}\right)=\frac{F_k^{(s)}}{\zeta_k(s)}, \tag{125}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1+\frac{n^l-1}{n^l(n^{s+1}-n^s-1)}\right)=\frac{A_k^{(s+l)}}{A_k^{(s)}}, \tag{126}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{n^l-1}{n^{s+l+1}-n^{s+l}-1}\right)=\frac{A_k^{(s)}}{A_k^{(s+l)}}, \tag{127}$$

$$\prod_{n\ge 2,\Omega(n)=k}\left(1-\frac{2n^{s+1}-1}{n^{2s}(n^2-1)}\right)=A_k^{(s)}Q_k^{(s)}, \tag{128}$$

$$\prod_{n\geq 2,\Omega(n)=k}\left(1-\frac{2}{n^{s+2}-n^s-n+1}\right)=\frac{A_k^{(s)}}{Q_k^{(s)}}, \tag{129}$$

$$\prod_{n\geq 2,\Omega(n)=k}\left(1+\frac{2}{n^{s+2}-n^s-n-1}\right)=\frac{Q_k^{(s)}}{A_k^{(s)}}, \tag{130}$$

$$\prod_{n\geq 2,\Omega(n)=k}\left(1+\frac{2n^{s+1}-n^s-2)}{n^{2s}(n-1)}\right)=A_k^{(s)}F_k^{(s)}, \tag{131}$$

$$\prod_{n\geq 2,\Omega(n)=k}\left(1-\frac{3n^{s+1}-2n^s-2)}{n^{2s+1}(n-1)}\right)=A_k^{(s)}F_k^{(s+1)}, \tag{132}$$

$$\prod_{n\geq 2,\Omega(n)=k}\left(1+\frac{2n-3}{n^{s+1}-n^s-2n+2}\right)=\frac{A_k^{(s)}}{F_k^{(s)}}, \tag{133}$$

$$\prod_{n\geq 2,\Omega(n)=k}\left(1+\frac{n+2}{n^{s+2}+n^{s+1}-2n-2}\right)=\frac{Q_k^{(s)}}{F_k^{(s+1)}}, \tag{134}$$

$$\prod_{n\geq 2,\Omega(n)=k}\left(1-\frac{n+2}{n(n^{s+1}+n^s-1)}\right)=\frac{F_k^{(s+1)}}{Q_k^{(s)}}, \tag{135}$$

$$\prod_{n\geq 2,\Omega(n)=k}\left(1-\frac{2n^l+2}{n^{s+l}-2}\right)=\frac{F_k^{(s)}}{F_k^{(s+l)}}. \tag{136}$$

Examples which involve $C_k^{(r)}$ have been left out for aesthetic reasons, as the dependence of the lower limit n on r in (86) leads to convoluted subcase notation, like

$$\prod_{n\geq 5,\Omega(n)=k}\left(1+\frac{3}{n(n-4)}\right)=\begin{cases}\frac{(27/16)C_k^{(3)}}{C_k^{(4)}}, & k=2,\\ \frac{C_k^{(3)}}{C_k^{(4)}}, & k\neq 2,\end{cases} \tag{137}$$

which notices that the semiprime $n=4$ contributes to $C_k^{(3)}$ with a factor $16/27$ if $k=2$, but not otherwise.

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